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International Journal of Advanced Research in Mathematics and Applications

Volume: 1 Issue: 2 Jul,2016,ISSN_NO: 2350-028X

On the solution set of the nonlocal problem of Itô stochastic differential equation

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Abstract

In this paper we are concerned with two Itô problems of stochastic differential equation with nonlocal condition, the solutions are represented as stochastic integral equations that contain Itô integral or in a special case mean square Riemann-Steltjes integral. We study the existence of at least mean square continuous solution for these types. The existence of the maximal and minimal solutions will be proved.

Keywords: Itô integral, mean square Riemann-Steltjes integral, Brownian motion, random Caratheodory function, stochastic Lebesgue dominated convergence theorem, at least mean square continuous solution, maximal solution, minimal solution.

1 Introduction

Problems of the stochastic differential equations have been extensively studied by several authors in the last decades, especially this type of Itô differential equation which leads to a Brownian motion as an integrator (Itô integral). A Brownian motion $W(t), t \in R$, is defined as a stochastic process such that

$$W(0) = 0,\ E(W(t)) = 0,\ \ E(W(t))^2 = t$$

We define Itô integral $\int_0^t g(s, X(s))dW(s)$ where the function $g(s, X(s))$ is a stochastic function of the parameter t and the second order stochastic process $X(t)$ and when

$$g(s, X(s)) = G(s)$$

Itô integral deleted into Riemann-Steltjes integral where the function $G(s)$ is only a continuous deterministic function. The reader is referred to ([1]-[2]), ([9]-[17]) and references therein.

Here our problem in stochastic are together with nonlocal conditions, The reader is referred to ([3]-[8]) and references therein for problems in ordinary differential equation with nonlocal conditions.

Here we are concerned with the stochastic differential equation

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) \tag{1}$$

with the random nonlocal initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \tag{2}$$

and the special case of equation (1)

$$dX(t) = f(t, X(t))dt + G(t)dW(t), \tag{3}$$

where X_0 is a second order random variable independent of the Brownian motion $W(t)$, a_k are positive real numbers and $G(t)$ is deterministic function.
The existence of at least mean square continuous solution, for each problem, will be studied.The existence of the maximal and minimal solutions will be proved.

2 Preliminaries

In our paper we need the following.

Definition 2.1 *[16][Random Caratheodory function]*
Let X be a stochastic process and let $t \in I$. A stochastic function $f(t, X(\omega))$ is called a Caratheodory function if it satisfies the following conditions

1. *$f(t, X(.))$ is measurable for every t,*
2. *$f(., X(\omega))$ is continuous for a.e. stochastic process X.*

Theorem 2.1 *[15][Schauder and Tychonoff theorem]*
Let Q be a closed, bounded convex set in a Banach space, and Let T be a completely continuous operator on Q such that $T(Q) \subset Q$. Then T has at least one fixed point in Q. That is, there is at least one $x^* \in Q$ such that $T(x^*) = x^*$.

Theorem 2.2 *[13][Arzela theorem]*
Every uniformly bounded equicontinuous family (sequence) of functions $(f_1(x), f_2(x), ..., f_k(x))$ has at least one subsequence which converges uniformly on the $I[a, b]$

Definition 2.2 *[13] A family of real random functions $(X_1(t), X_2(t), ..., X_k(t))$ is uniformly bounded in mean square sense if there exist a $\beta \in R$ (β is finite) such that*

$$E(X_n^2(t)) < \beta.$$

for all $n \geq 1$ and all $t \in T$.

Definition 2.3 *[13] A family of real random functions $(X_1(t), X_2(t), ..., X_k(t))$ is equicontinuous in mean square sense if for each $t \in T$ and $\epsilon > 0$, there exist a $\delta > 0$ such that*

$$E([X_n(t_2) - X_n(t_1)]^2) < \epsilon.$$

for all $n \geq 1$ when ever $\mid t_2 - t_1 \mid < \delta$.

Theorem 2.3 *[14][Stochastic Lebesgue dominated convergence theorem]*
Let $X_n(t)$ be a sequence of random vectors (or functions) is converging to $X(t)$ such that

$$X(t) = \lim_{n\to\infty} X_n(t), \quad a.s.,$$

and $X_n(t)$ is dominated by an integrable function $a(t)$ such that

$$\| X_n(t) \|_2 \leq a(t).$$

Then

1. $E[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} E[X_n]$ *and*
2. $E[X_n(t) - X(t)] \to 0 \quad as \quad n \to \infty$

3 Integral equation representation

Let $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic process with the norm

$$\| X \|_C = \sup_{t\in[0,T]} \| X(t) \|_2 = \sup_{t\in[0,T]} \sqrt{E\,(X(t))^2}.$$

Throughout the paper we assume that the following assumptions hold

i- The functions $f, g : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ are Caratheodory function in mean square sense.

ii- There exists two integrable functions $l(t)$ and $h(t) \in L^1$ such that

$$\| f(t,X) \|_2 \leq l(t)\ , \ \ \| g(t,X) \|_2 \leq h(t), \quad \forall (t,X) \in I \times L_2(\Omega)$$

and

$$\int_{t_1}^{t_2} l(t) \leq k_1, \ \ \int_{t_1}^{t_2} h^2(t) \leq k_2, \quad \forall t \in I$$

Now we have the following lemma.

Lemma 3.1 *The solution of the stochastic nonlocal problem (1) and (2) can be expressed by the integral equation*

$$\begin{aligned} X(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s,X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s,X(s))dW(s)\right) \\ &+ \int_0^t f(s,X(s))ds + \int_0^t g(s,X(s))dW(s) \end{aligned} \tag{4}$$

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where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$.

Proof. Integrating equation (1), we obtain

$$X(t) = X(0) + \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dW(s),$$

and

$$X(\tau_k) = X(0) + \int_0^{\tau_k} f(s, X(s))ds + \int_0^{\tau_k} g(s, X(s))dW(s),$$

then

$$\sum_{k=1}^{n} a_k X(\tau_k) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} \int_0^{\tau_k} g(s, X(s))dW(s),$$

$$X_0 - X(0) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} \int_0^{\tau_k} g(s, X(s))dW(s),$$

and

$$\left(1 + \sum_{k=1}^{n} a_k\right) X(0) = X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} \int_0^{\tau_k} g(s, X(s))dW(s),$$

then

$$X(0) = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1} \left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s, X(s))dW(s)\right).$$

Hence

$$\begin{aligned} X(t) &= a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s, X(s))dW(s)\right) \\ &+ \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dW(s), \end{aligned}$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$.■

Consider now the following assumptions

A1- The function $f : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ is Caratheodory function in mean square sense.

A2- There exists an integrable function $l(t) \in L^1$ such that

$$\| f(t,X) \|_2 \leq l(t) \qquad \forall (t,X) \in I \times L_2(\Omega)$$

and

$$\int_{t_1}^{t_2} l(t)dt \leq k, \quad \forall t \in I$$

A3- $G : R^+ \to R^+$ is a continuous deterministic function.

By the same way of Lemma [3.1] we can prove the following lemma.

Lemma 3.2 *Let the assumptions (A1)-(A3) be satisfied. The solution of the nonlocal stochastic problem (3) and (2) can be expressed by the integral equation*

$$X(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s,X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} G(s)dW(s)\right) + \int_0^t f(s,X(s))ds + \int_0^t G(s)dW(s), \quad (5)$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$.

4 Existence of at least mean square continuous solution

For the existence of at least continuous solution $X \in C$ of the Itô problem (1) and (2), we have the following theorem.

Theorem 4.4 *Let the assumptions (i)-(ii) be satisfied, then the problem (1)-(2) has at least one solution* $X \in C$ *given by the stochastic integral equation (4).*

Proof. Consider in the space C, the set Q such that

$$Q = \{X \in C : \| X \|_C \leq \beta\}$$

Now for each $X \in Q$ we can define the operator H by

$$HX(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s,X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s,X(s))dW(s)\right) + \int_0^t f(s,X(s))ds + \int_0^t g(s,X(s))dW(s)$$

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International Journal of Advanced Research in Mathematics and Applications

Volume: 1 Issue: 2 Jul,2016,ISSN_NO: 2350-028X

we shall prove that $H : Q \to Q$. For that let $X(t) \in Q$, then

$$\begin{aligned}
\| HX(t) \|_2 &\leq a \| X_0 \|_2 + a \sum_{k=1}^{n} a_k \int_0^{\tau_k} \| f(s, X(s)) \|_2 \, ds + a \sum_{k=1}^{n} a_k \left\| \int_0^{\tau_k} g(s, X(s)) dW(s) \right\|_2 \\
&+ \int_0^t \| f(s, X(s)) \|_2 \, ds + \left\| \int_0^t g(s, X(s)) dW(s) \right\|_2 \\
&\leq a \| X_0 \|_2 + a \sum_{k=1}^{n} a_k \int_0^{\tau_k} l(s) ds + a \sum_{k=1}^{n} a_k \int_0^{\tau_k} h^2(s) ds + \int_0^t l(s) ds + \int_0^t h^2(s) ds \\
&\leq a \| X_0 \|_2 + a \sum_{k=1}^{n} a_k k_1 + a \sum_{k=1}^{n} a_k k_2 + k_1 + k_2 \\
&= \beta
\end{aligned}$$

then ($\| HX \|_C \leq \beta$), so $HX \in Q$ and hence $HQ \subset Q$ and is also uniformly bounded. For $t_1, t_2 \in R^+$, $t_1 < t_2$, let $| t_2 - t_1 | < \delta$, then

$$\begin{aligned}
\| HX(t_2) - HX(t_1) \|_2 &\leq \int_{t_1}^{t_2} \| f(s, X(s)) \|_2 \, ds + \left\| \int_{t_1}^{t_2} g(s, X(s)) dW(s) \right\|_2 \\
&\leq \int_{t_1}^{t_2} l(s) ds + \int_{t_1}^{t_2} h^2(s) ds \\
&\leq k_1 + k_2, \\
&\leq k,
\end{aligned}$$

where $k = \sup\{k_1, k_2\}$.
Then $\{HX\}$ is a class of equicontinuous functions. Therefore the operator H is equicontinuous and uniformly bounded.
Suppose that $\{X_n\} \in C$ such that $X_n \to X$ with probability 1

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So,

$$
\begin{aligned}
\underset{n\to\infty}{l.i.m}\ HX_n(t) &= \underset{n\to\infty}{l.i.m}\left[aX_0 - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s,X_n(s))ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s,X_n(s))dW(s)\right] \\
&+ \underset{n\to\infty}{l.i.m}\left[\int_0^{t} f(s,X_n(s))ds + \int_0^{t} g(s,X_n(s))dW(s)\right] \\
&= aX_0 - \left(a\sum_{k=1}^{n} a_k\right)\underset{n\to\infty}{l.i.m}\left[\int_0^{\tau_k} f(s,X_n(s))ds\right] \\
&- \left(a\sum_{k=1}^{n} a_k\right)\underset{n\to\infty}{l.i.m}\left[\int_0^{\tau_k} g(s,X_n(s))dW(s)\right] \\
&+ \underset{n\to\infty}{l.i.m}\left[\int_0^{t} f(s,X_n(s))ds\right] + \underset{n\to\infty}{l.i.m}\left[\int_0^{t} g(s,X_n(s))dW(s)\right]
\end{aligned}
$$

Using our assumptions and then applying stochastic Lebesgue dominated convergence theorem, we get

$$
\begin{aligned}
\underset{n\to\infty}{l.i.m}\ HX_n(t) &= aX_0 - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \underset{n\to\infty}{l.i.m}\,[f(s,X_n(s))]ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \underset{n\to\infty}{l.i.m}\,[g(s,X_n(s))]dW(s) \\
&+ \int_0^{t} \underset{n\to\infty}{l.i.m}\,[f(s,X_n(s))]ds + \int_0^{t} \underset{n\to\infty}{l.i.m}\,[g(s,X_n(s))]dW(s)
\end{aligned}
$$

$$
\begin{aligned}
&= aX_0 - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} [f(s,\underset{n\to\infty}{l.i.m}\,X_n(s))]ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} [g(s,\underset{n\to\infty}{l.i.m}\,X_n(s))]dW(s) \\
&+ \int_0^{t} [f(s,\underset{n\to\infty}{l.i.m}\,X_n(s))]ds + \int_0^{t} [g(s,\underset{n\to\infty}{l.i.m}\,X_n(s))]dW(s) \\
&= aX_0 - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s,X(s))ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s,X(s))dW(s) \\
&+ \int_0^{t} f(s,x(s))ds + \int_0^{t} g(s,x(s))dW(s) \\
&= HX(t)
\end{aligned}
$$

This proves that H is continuous operator, then H is continuous and compact.

Then H has a fixed point $X \in C$ which proves that there exists at least one solution of the stochastic differential equation (1)-(2) given by (4).■
Now we have the following theorem.

Theorem 4.5 *Let the assumptions (A1)-(A3) be satisfied, then the problem (3) and (2) has at least mean square continuous solution $X \in C$ given by the stochastic integral equation (5).*

5 Maximal and minimal solution

Now we give the following definition

Definition 5.4 *Let $q(t)$ be a solution of the problem (1)-(2), then $q(t)$ is said to be a maximal solution of (1)-(2) if every solution $X(t)$ of (1)-(2) satisfies the inequality*

$$E(X^2(t)) < E(q^2(t)).$$

A minimal solution $s(t)$ can be defined by similar way by reversing the above inequality i.e.

$$E(X^2(t)) > E(s^2(t)).$$

In this section f assumed to satisfy the following definition.

Definition 5.5 *The function $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically decreasing if for any $X, Y \in L_2(\Omega)$ satisfying*

$$\| X(t) \|_2 < \| Y(t) \|_2 .$$

implies that

$$\| f(t, X(t)) \|_2 < \| f(t, Y(t)) \|_2 .$$

Now we have the following lemma

Lemma 5.3 *let the assumptions (i)-(ii) be satisfied and let $X, Y \in C$ satisfying*

$$\begin{aligned} \| X(t) \|_2 \;\; &\leq \;\; a\left(\| X_0 \|_2 + \sum_{k=1}^{n} a_k \int_0^{\tau_k} \| f(s, X(s)) \|_2 \, ds + \sum_{k=1}^{n} a_k \left\| \int_0^{\tau_k} g(s, X(s)) dW(s) \right\|_2 \right) \\ &+ \int_0^t \| f(s, X(s)) \|_2 \, ds + \left\| \int_0^t g(s, X(s)) dW(s) \right\|_2 , \end{aligned}$$

and

$$\begin{aligned} \| Y(t) \|_2 \;\; &\geq \;\; a\left(\| X_0 \|_2 + \sum_{k=1}^{n} a_k \int_0^{\tau_k} \| f(s, Y(s)) \|_2 \, ds + \sum_{k=1}^{n} a_k \left\| \int_0^{\tau_k} g(s, Y(s)) dW(s) \right\|_2 \right) \\ &+ \int_0^t \| f(s, Y(s)) \|_2 \, ds + \left\| \int_0^t g(s, Y(s)) dW(s) \right\|_2 , \end{aligned}$$

If $f(t;x)$ and $g(t,x)$ are stochastically decreasing function . Then

$$\| X(t) \|_2 < \| Y(t) \|_2 \tag{6}$$

Proof. Let the conclusion (6) be false, then there exists t_1 such that

$$\| X(t_1) \|_2 = \| Y(t_1) \|_2, \quad t_1 > 0 \tag{7}$$

and

$$\| X(t) \|_2 < \| Y(t) \|_2, \quad 0 < t < t_1 \tag{8}$$

since $f(t;X)$ and $g(t,X)$ satisfy the definition (5.5)and using equation (8), we get

$$\begin{aligned}
\| X(t_1) \|_2 &\leq a\left(\| X_0 \|_2 + \sum_{k=1}^{n} a_k \int_0^{\tau_k} \| f(s,X(s)) \|_2 \, ds + \sum_{k=1}^{n} a_k \left\| \int_0^{\tau_k} g(s,X(s))dW(s) \right\|_2 \right) \\
&+ \int_0^{t_1} \| f(s,X(s)) \|_2 \, ds + \left\| \int_0^{t_1} g(s,X(s))dW(s) \right\|_2 , \\
&< a\left(\| X_0 \|_2 + \sum_{k=1}^{n} a_k \int_0^{\tau_k} \| f(s,Y(s)) \|_2 \, ds + \sum_{k=1}^{n} a_k \left\| \int_0^{\tau_k} g(s,Y(s))dW(s) \right\|_2 \right) \\
&+ \int_0^{t_1} \| f(s,Y(s)) \|_2 \, ds + \left\| \int_0^{t_1} g(s,Y(s))dW(s) \right\|_2 , \\
&< \| Y(t) \|_2, \quad 0 < t < t_1
\end{aligned}$$

Which contradicts the fact assumed in equation (7), then

$$\| X(t) \|_2 < \| Y(t) \|_2 . \blacksquare$$

now we have the following theorem.

Theorem 5.6 *Let the assumptions (i)-(ii) be satisfied. If $f(t,X)$ and $g(t,X)$ satisfy the definition (5.5), then there exist a maximal solution of problem (1)-(2).*

Proof. Firstly we shall prove the existence of the maximal solution of the problem. Let $\epsilon > 0$ be given. Now consider the integral equation

$$\begin{aligned}
X_\epsilon(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_\epsilon(s,X_\epsilon(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g_\epsilon(s,X_\epsilon(s))dW(s) \right) \\
&+ \int_0^{t} f_\epsilon(s,X_\epsilon(s))ds + \int_0^{t} g_\epsilon(s,X_\epsilon(s))dW(s),
\end{aligned} \tag{9}$$

where

$$f_\epsilon(t, X_\epsilon(t)) = f(s, X_\epsilon(t)) + \epsilon$$

and

$$g_\epsilon(t, X_\epsilon(t)) = g(s, X_\epsilon(t)) + \epsilon$$

Clearly the function $f_\epsilon(t, X_\epsilon(t))$ and $g_\epsilon(t, X_\epsilon(t))$ satisfies the conditions (i)-(ii) and

$$\| f_\epsilon(t, X_\epsilon(t)) \|_2 \leq l(t) + \epsilon = \grave{l}(t)$$

also

$$\| g_\epsilon(t, X_\epsilon(t)) \|_2 \leq h(t) + \epsilon = \grave{h}(t)$$

then equation (9) is a solution of the problem (1)-(2) according to theorem (4.4)
Now let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$ Then

$$\begin{aligned} X_{\epsilon_1}(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_{\epsilon_1}(s, X_{\epsilon_1}(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g_{\epsilon_1}(s, X_{\epsilon_1}(s))dW(s)\right) \\ &+ \int_0^t f_{\epsilon_1}(s, X_{\epsilon_1}(s))ds + \int_0^t g_{\epsilon_1}(s, X_{\epsilon_1}(s))dW(s), \\ &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (f(s, X_{\epsilon_1}(s)) + \epsilon_1)\, ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (g(s, X_{\epsilon_1}(s)) + \epsilon_1)\, dW(s)\right) \\ &+ \int_0^t (f(s, X_{\epsilon_1}(s)) + \epsilon_1)\, ds + \int_0^t (g(s, X_{\epsilon_1}(s)) + \epsilon_1)\, dW(s), \end{aligned}$$

this implies that

$$\begin{aligned} aX_0 - X_{\epsilon_1}(t) &= a\sum_{k=1}^{m} a_k \int_0^{\tau_k} (f(s, X_{\epsilon_1}(s)) + \epsilon_1)\, ds + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} (g(s, X_{\epsilon_1}(s)) + \epsilon_1)\, dW(s) \\ &- \int_0^t (f(s, X_{\epsilon_1}(s)) + \epsilon_1)\, ds - \int_0^t (g(s, X_{\epsilon_1}(s)) + \epsilon_1)\, dW(s), \end{aligned}$$

since we have

$$a \| X_0 \|_2 - \| X_{\epsilon_1}(t) \|_2 \leq \| aX_0 - X_{\epsilon_1}(t) \|_2$$

so

$$\begin{aligned} a \| X_0 \|_2 - \| X_{\epsilon_1}(t) \|_2 &\leq a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||(f(s, X_{\epsilon_1}(s)) + \epsilon_2)||_2\, ds + a\sum_{k=1}^{m} a_k \left|\left|\int_0^{\tau_k} (g(s, X_{\epsilon_1}(s)) + \epsilon_2)\, dW(s)\right|\right|_2 \\ &+ \int_0^t ||(f(s, X_{\epsilon_1}(s)) + \epsilon_2)||_2\, ds + \left|\left|\int_0^t (g(s, X_{\epsilon_1}(s)) + \epsilon_2)\, dW(s)\right|\right|_2, \end{aligned}$$

then

$$\begin{aligned}\| X_{\epsilon_1}(t) \|_2 &\geq a \| X_0 \|_2 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||(f(s, X_{\epsilon_1}(s)) + \epsilon_2)||_2 \, ds - a\sum_{k=1}^{m} a_k \left\| \int_0^{\tau_k} (g(s, X_{\epsilon_1}(s)) + \epsilon_2) \, dW(s) \right\|_2 \\ &- \int_0^t ||(f(s, X_{\epsilon_1}(s)) + \epsilon_2)||_2 \, ds - \left\| \int_0^t (g(s, X_{\epsilon_1}(s)) + \epsilon_2) \, dW(s) \right\|_2 , \end{aligned} \tag{10}$$

and

$$\begin{aligned}\| X_{\epsilon_2}(t) \|_2 &\leq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||(f(s, X_{\epsilon_2}(s)) + \epsilon_2)||_2 \, ds + a\sum_{k=1}^{m} a_k \left\| \int_0^{\tau_k} (g(s, X_{\epsilon_2}(s)) + \epsilon_2) \, dW(s) \right\|_2 \\ &+ \int_0^t ||(f(s, X_{\epsilon_2}(s)) + \epsilon_2)||_2 \, ds + \left\| \int_0^t (g(s, X_{\epsilon_2}(s)) + \epsilon_2) \, dW(s) \right\|_2 , \end{aligned} \tag{11}$$

Using lemma 5.3, then equations (10) and (11) implies

$$\| X(\epsilon_2(t)) \|_2 < \| X(\epsilon_1(t)) \|_2$$

As shown before in the proof of theorem (4.4) the family of functions $x_\epsilon(t)$ defined by equation (4) is uniformly bounded and equi-continuous functions. Hence by Arzela-Ascoli Theorem , there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} X_{\epsilon_n}(t)$ exists uniformly in C and denote this limit by $q(t)$ from the continuity of the function f_{ϵ_n} in the second argument and applying Lebesgue Dominated Convergence Theorem , we get

$$q(t) = \lim_{n\to\infty} X_{\epsilon_n}(t)$$

which proves that $q(t)$ as a solution of the problem (1)-(2)
Finally, we shall show that $q(t)$ is the maximal solution of the problem (1)-(2). To do this, let $X(t)$ be any solution of the problem (1)-(2). Then

$$\begin{aligned}\| X_{\epsilon}(t) \|_2 &\geq a \| X_0 \|_2 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||(f(s, X_{\epsilon}(s)) + \epsilon)||_2 \, ds - a\sum_{k=1}^{m} a_k \left\| \int_0^{\tau_k} (g(s, X_{\epsilon}(s)) + \epsilon) \, dW(s) \right\|_2 \\ &- \int_0^t ||(f(s, X_{\epsilon}(s)) + \epsilon)||_2 \, ds - \left\| \int_0^t (g(s, X_{\epsilon}(s)) + \epsilon) \, dW(s) \right\|_2 , \end{aligned}$$

and

$$\begin{aligned}\| X(t) \|_2 \;\leq\; & a\left(||X_0||_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} ||f(s, X(s))||_2\, ds + \sum_{k=1}^{m} a_k \left\| \int_0^{\tau_k} g(s, X(s)) dW(s) \right\|_2 \right) \\ & + \int_0^t ||f(s, X(s))||_2\, ds + \left\| \int_0^t g_\epsilon(s, X(s)) dW(s) \right\|_2 ,\end{aligned}$$

Applying lemma (5.3), we get

$$\| X_\epsilon(t) \|_2 > \| X(t) \|_2$$

from the uniqueness of the maximal solution (see [6]), it is clear that $X_\epsilon(t)$ tends to $q(t)$ uniformly as $\epsilon \to 0$. ■

By similar way as done above we can prove that $s(t)$ is the minimal solution of the problem (1)-(2). The maximal and minimal solutions of the problem (1)-(2) can be defined in the same fashion as done above.

If the functions f and g assumed to satisfy the following definition.

Definition 5.6 *The function $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically increasing if for any $X, Y \in L_2(\Omega)$ satisfying*

$$\| X(t) \|_2 < \| Y(t) \|_2 .$$

implies that

$$\| f(t, X(t)) \|_2 > \| f(t, Y(t)) \|_2 .$$

and we have the following theorem.

Theorem 5.7 *Let the assumptions (i)-(ii) be satisfied. If $f(t, X)$ and $g(t, X)$ satisfy the definition (5.6), then there exist a minimal solution of the problem (1)-(2).*

Now we have the following theorem.

Theorem 5.8 *Let the assumptions of theorem (4.5) be satisfied. Then there exist a maximal and minimal solutions of the problem (3)-(2).*

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